Generation of Gravitational Radiation in the Laboratory

F. Romero B. and H. Dehnen
Fakultät für Physik der Universität Konstanz

Z. Naturforsch. **36a**, 948—955 (1981); received July 9, 1981

The generation of gravitational radiation by coherent excitation of a long row of oscillators is investigated. In particular we have considered diatomic linear chains as an idealization of thin piezoelectric crystals. We find a highly focussed superradiant beam of gravitational radiation in direction of the row and a total radiation power larger than the incoherent superposition of the oscillator radiation by the factor λ/a (λ wavelength*, a distance between neighbouring oscillators). In spite of these optimum conditions it seems to us that the attainable magnitude of the radiation power of approximately 10^{-22} erg/sec is not high enough for a successful laboratory experiment.

1. Introduction

From the beginning of the "gravitational radiation era", based principally on the pioneering work of Weber [1], one was almost exclusively dedicated to develop appropriate antennae for the *detection* of the theoretically predicted gravitational waves coming from celestial bodies. We can not say certainly until now that a *direct* detection of this radiation in the laboratory succeeded. On the other hand only few authors, see e.g. [2—4], have given estimations about the possibility of laboratory experiments for the *generation* and subsequent detection of gravitational waves. However these results have not found a consensus until now (see e.g. [5], [6]). For this reason we intend to show in this paper by means of a particular arrangement of oscillators that the generation in the laboratory is really a hard task even in case of most favourable conditions. In comparison with other authors we do not assume any special excitation mechanism for our arrangement, leaving this decision to the experimentators; we just demand an optimum on-phase coherent excitation which should give a highly focussed superradiant beam of gravitational radiation and a kind of stimulated emission. Another essential difference between the mentioned works and ours is that we choose for our investigation a fully classsical way.

We perform the whole calculation within the linear theory of gravitation and give at first a brief summary of the most important results used in the work. Assuming for the space-time metric the weak-field approximation ($\eta^{\mu\nu} = \mathrm{diag}\,(-1, +1, +1, +1)$):

$$g^{\mu\nu} = \eta^{\mu\nu} + h^{\mu\nu}, \quad |h^{\mu\nu}| \ll 1, \tag{1.1}$$

with the gauge condition

$$\bar{h}^{\mu\nu}_{\ |\nu} = 0, \quad \bar{h}^{\mu\nu} = h^{\mu\nu} - \tfrac{1}{2} h\, \eta^{\mu\nu}, \quad (h \equiv h^{\mu\nu}\eta_{\mu\nu}) \tag{1.2}$$

the field equations read ($G = 1$, $c = 1$)

$$\Box\, \bar{h}^{\mu\nu} = -16\pi T^{\mu\nu}. \tag{1.3}$$

The well known far-field solution of (1.3) (see for example [7]) is

$$\bar{h}^{\mu\nu}(t, \boldsymbol{x}) = \frac{4}{r} \int T^{\mu\nu}(t - |\boldsymbol{x} - \boldsymbol{x}'|, \boldsymbol{x}')\, \mathrm{d}^3x', \tag{1.4}$$

whose space-like components** can be set within the "quadrupole formalism" into the form

$$\bar{h}^{jk}(t, \boldsymbol{x}) = \frac{2}{r} \frac{\mathrm{d}^2}{\mathrm{d}t^2} \int \varrho(t', \boldsymbol{x}')\, x'^j x'^k\, \mathrm{d}^3x', \tag{1.4a}$$

where $t' \simeq t - r$ is the retarded time between field point and center of mass of the source (ϱ mass-density). This quadrupole approximation is only valid for the low-frequency limit ($\lambda \gg L$, L linear dimension of the source), which implies slow-motions within the source.

For the energy flux of the radiation in radial direction we have [8]

$$T^{0r}_{\mathrm{GW}} = \frac{1}{32\pi} \langle \bar{h}^{\mathrm{TT}}_{jk|0}\, \bar{h}^{\mathrm{TT}}_{jk|0} \rangle \tag{1.5}$$

(bracket means average over several wave lengths),

Reprint requests to Prof. Dr. H. Dehnen, Fakultät für Physik der Universität Konstanz, Postfach 5560, D-7750 Konstanz.

* *Anm. der Redaktion*: Aus satztechnischen Gründen wurde λ statt $\lambda\!\!\!^{-}$ gesetzt.

** Latin indices run from 1 to 3, greek indices from 0 to 3.

0340-4811 / 81 / 0900-0948 $ 01.00/0. — Please order a reprint rather than making your own copy.

wherein

$$\bar{h}^{TT}_{jk} = P_{jl}P_{mk}\bar{h}_{lm} - \tfrac{1}{2}P_{jk}(P_{ml}\bar{h}_{lm}) \qquad (1.6)$$

represents the transverse and traceless projection of the metric perturbation performed by the projection tensor $P_{jk} = \delta_{jk} - n_j n_k$, which projects on the 2-dimensional plane orthogonal to the propagation direction of the wave $n_k = k_k/|\boldsymbol{k}|$ (k_k 3-dimensional wave vector). In this projection $\bar{h}^{TT}_{jk}$ coincides with h^{TT}_{jk}.

2. The Model of the Source and its Radiation Field

We describe in the following a source for gravitational radiation whose excitation allows a type of "superradiance". Such a model is represented by a row of identical oscillators (two-mass vibrators), which are ordered on a line with constant distance with respect to one another. The vibrators have their axes mutually parallel and are located orthogonal to the row axis, which is chosen identical with the y-axis. The arrangement and the allowed displacements of the single vibrators are shown in Figs. 1 and 2, respectively, whereby the single vibrator has the following characteristics: two equal masses M coupled harmonically by a spring and separated by a distance $2b$ at the equilibrium position. The distance between two neighbouring vibrators is a and the row begins at $y = 0$ and finishes at $y = (N-1)a$, where N is the total number of vibrators in the row.

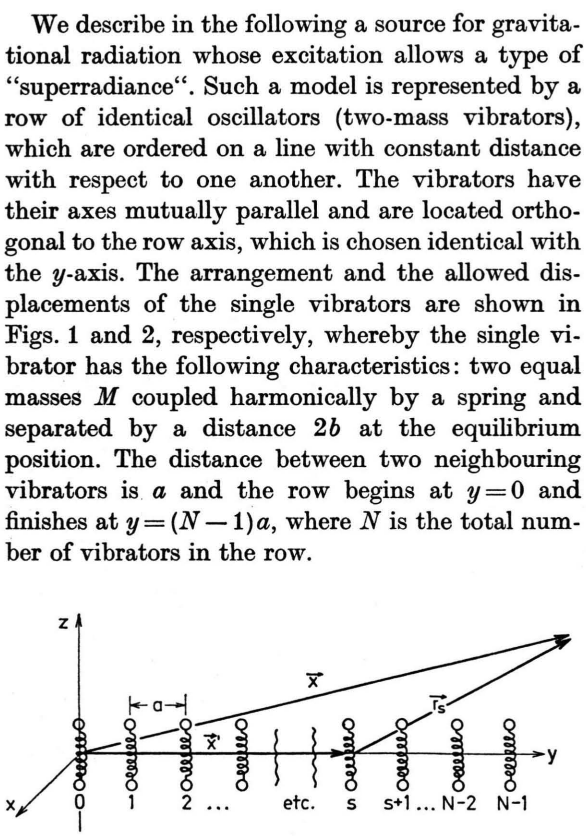

Fig. 1. The whole row of vibrators.

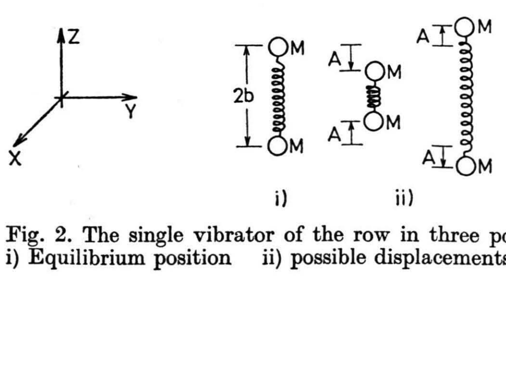

Fig. 2. The single vibrator of the row in three positions. i) Equilibrium position ii) possible displacements.

Next we calculate the gravitational radiation of this device, whereby an adequate on-phase excitation of the elements is assumed in order to obtain a highly focussed radiation of the antenna and a certain superradiance in the beam. The displacements out of the equilibrium position for the masses of the s-th vibrator take the form

$$u^s(t') = \pm\,(b + A\sin[\Omega(t' + \Phi(s))]) \qquad (2.1)$$

(Ω frequency, A amplitude and Φ phase). The phase $\Phi(s)$ will be determined later in such a way that a positive superposition of the emitted radiation in the direction of the y-axis results. Then the mass distribution along the row reads

$$\varrho = M\sum_{s=0}^{N-1}\{\delta^3(\boldsymbol{x}' - [s\,a\,\boldsymbol{e}_y + u^s(t')\,\boldsymbol{e}_z]) + \delta^3(\boldsymbol{x}' - [s\,a\,\boldsymbol{e}_y - u^s(t')\,\boldsymbol{e}_z])\} \qquad (2.2)$$

with $\boldsymbol{e}_y$ and $\boldsymbol{e}_z$ unit vectors in direction of the y- and z-axis, respectively, wherein we have considered that the vibrators are located parallel to the z-axis with their centers of mass on the y-axis. For the calculation of the gravitational radiation we consider frequencies Ω, such that the wave length of the emitted radiation is larger than the linear dimension of the vibrator, i.e. $\lambda\!\!\!^{-} \gg b$, but arbitrary with respect to the length of the row. Since the single vibrator can be considered as a closed system ($T^{\nu}_{\mu|\nu} = 0$), which interacts with the neighbouring system only through the freely disposable phase Φ, it is justified to use the quadrupole formalism mentioned above for each single vibrator and to superpose subsequently the wave amplitudes produced by all vibrators of the row in a coherent way (for an alternative procedure of obtaining the total radiation field see for example [9]).

According to (1.6) and (1.4a) the radiation field of the s-th vibrator reads

$$\overset{s}{h}{}^{TT}_{jk} = \frac{2}{r_s}\,\ddot{\underline{I}}{}^{TT}_{jk}(t - r_s + \Phi(s))\,, \qquad (2.3)$$

where

$$\underline{I}_{jk} = I_{jk} - \tfrac{1}{3}\delta_{jk}I\,,\quad I = I_{jk}\delta_{jk}\,,$$
$$I_{jk} \equiv \int \overset{s}{\varrho}(\boldsymbol{x}', t')\,x'_j x'_k\,\mathrm{d}^3x' \qquad (2.3\mathrm{a})$$

is the mass-quadrupole tensor and r_s the distance from the center of mass of the s-th vibrator to the field point. The mass distribution for the single vibrators is given by the single terms of (2.2) as

$$\overset{s}{\varrho} = M\{\delta^3(\boldsymbol{x}' - [s\,a\,\boldsymbol{e}_y + u^s(t')\,\boldsymbol{e}_z]) + \delta^3(\boldsymbol{x}' - [s\,a\,\boldsymbol{e}_y - u^s(t')\,\boldsymbol{e}_z])\}\,. \qquad (2.4)$$

For the whole vibrator row the radiation field results in

$$h_{jk}^{\mathrm{TT}} = \sum_{s=0}^{N-1} \overset{s}{h}{}_{jk}^{\mathrm{TT}}. \tag{2.5}$$

Inserting (2.3) into (2.5) we get

$$h_{jk}^{\mathrm{TT}} = \frac{2}{r} \sum_{s=0}^{N-1} \ddot{I}_{jk}^{\mathrm{TT}}(t - r_s + \Phi(s)), \tag{2.6}$$

where $r_s \simeq r - s a \sin\theta \sin\varphi$ (θ, φ usual polar angles). The demand for constructive superposition of the radiation of the row in direction of the y-axis requires for the phase-shift $\Phi(s) = -sa$. Herewith and with (2.4) one obtains from (2.3a) for small vibration amplitudes ($A \ll 2b$) after projection according to (1.6)

$$\begin{aligned}\ddot{I}_{jk}^{\mathrm{TT}} &= -4 M A b \Omega^2 \\ &\quad\cdot \sin[\Omega(t - r + s a(\sin\theta\sin\varphi - 1))] \\ &\quad\cdot \frac{(1 - n_z{}^2)}{2} e_{jk}.\end{aligned} \tag{2.7}$$

Insertion into (2.6) gives

$$\begin{aligned}h_{jk}^{\mathrm{TT}} &= \frac{-8 M b A \Omega^2}{r} \frac{(1 - n_z{}^2)}{2} \\ &\quad\cdot \sum_{s=0}^{N-1} \{\sin[\Omega(t - r + s a(\sin\theta\sin\varphi - 1))]\} e_{jk}\end{aligned} \tag{2.8}$$

for the total radiation field of the row, where e_{jk} is the polarization tensor defined by

$$\begin{aligned}e_{jk} &= \frac{2}{\sin^2\theta} \{(\delta_{jz} - n_j \cos\theta)(\delta_{kz} - n_k\cos\theta) \\ &\quad - \tfrac{1}{2}\sin^2\theta(\delta_{jk} - n_j n_k)\}\end{aligned} \tag{2.9}$$

with the following general properties:

$$e_{jk} n^k = 0, \quad e_j{}^j = 0, \quad e_{jk} e^{jk} = 2. \tag{2.9a}$$

For the beam in direction of the y-axis it takes the simple form

$$e_{jk} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{2.9b}$$

Summing up the terms in (2.8) we get

$$\begin{aligned}h_{jk}^{\mathrm{TT}} &= \frac{-8 M b A \Omega^2}{r} \frac{(1 - n_z{}^2)}{2} \frac{\sin\left[N \dfrac{\Omega a}{2}(n_y - 1)\right]}{\sin\left[\dfrac{\Omega a}{2}(n_y - 1)\right]} \\ &\quad\cdot \{\sin[\Omega(t - r + (N-1)(a/2)(n_y - 1))]\}.\end{aligned} \tag{2.8a}$$

Herewith we obtain from (1.5) for the energy flux in radial direction:

$$\begin{aligned}T_{\mathrm{GW}}^{0r} &= \frac{1}{2\pi} \frac{(M b A)^2}{r^2} \\ &\quad\cdot \Omega^6 \frac{\sin^2\left[N \dfrac{\Omega a}{2}(n_y - 1)\right]}{\sin^2\left[\dfrac{\Omega a}{2}(n_y - 1)\right]} (1 - n_z{}^2)^2.\end{aligned} \tag{2.10}$$

This expression has the following properties of interest:

a) In case that $N = 1$ (the row is reduced to a single vibrator) it becomes independent of φ; the resulting expression agrees exactly with the energy flux for a vibrator (compare for example [10]) with its well-known (φ independent) angular distribution

$$T_{\mathrm{GW}}^{0r} = \frac{1}{2\pi} \frac{(M b A)^2}{r^2} \Omega^6 \sin^4\theta \tag{2.11a}$$

and the total radiation power

$$\begin{aligned}L_{\mathrm{GW}} &= \int T_{\mathrm{GW}}^{0r} r^2 \sin\theta \,\mathrm{d}\theta\,\mathrm{d}\varphi \\ &= \frac{16}{15}(M b A)^2 \Omega^6 \triangleq \frac{16}{15}\frac{G}{c^5}(M b A)^2 \Omega^6.\end{aligned} \tag{2.11b}$$

b) In case of several vibrators the expression (2.10) shows a strong direction dependence for the emission of the generated gravitational waves; for $\lambda\!\!\!^{-} \gg 2a/\pi$ (only one zero-point of the denominator in (2.10)) we find a very strong emission in direction of the y-axis and a minimum or vanishing emission in all other directions including the negative y-direction (see Figure 3). Consequently we have a typical focusing of the radiation, which possesses additionally a superradiant behaviour (radiation intensity $\sim N^2$); for the beam in direction of the y-axis it follows:

$$\begin{aligned}T_{\mathrm{GW}}^{0y} &= \frac{1}{2\pi} \frac{(M b A)^2}{r^2} \Omega^6 N^2 \\ &\triangleq \frac{1}{2\pi} \frac{G}{c^5} \frac{(M b A)^2}{r^2} \Omega^6 N^2.\end{aligned} \tag{2.10a}$$

In view of this superradiance the possibility of a laboratory experiment should be investigated. First, however, a more realistic description of the device is necessary. This and an estimation of the obtainable radiation power are given in the two next sections.

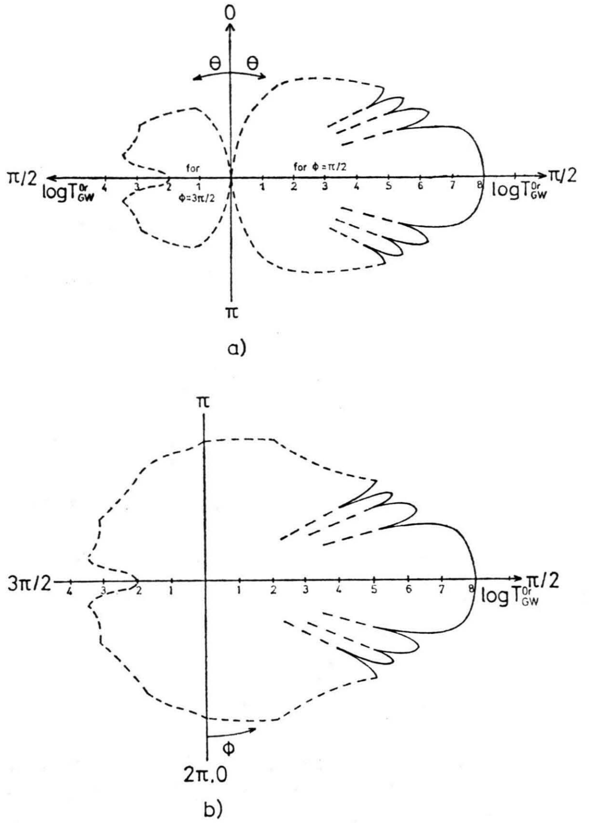

Fig. 3. Angular distribution for the gravitational radiation of the vibrator-row (formula (2.10) with $N=10^4$, $\Omega=10^9$ sec^{-1} and $a=0.5$ cm; the values for log T^{0r}_{GW} correspond alone to the angular part of T^{0r}_{GW}): a) $\varphi=\left\{\begin{smallmatrix}\pi/2\\ 3\pi/2\end{smallmatrix}\right.$, θ runs b) $\theta=\pi/2$, φ runs. Evidently there exists a "needle radiation" in direction of the y-axis ($\theta=\varphi=\pi/2$). This becomes more pronounced for increasing values of the product $\Omega\, a\, N$, see (2.12a).

Finally we analyse the properties of the total radiation power obtained by integration of (2.10) over the total sphere; for this we confine ourselves to the case $N\gg 1$ and $\underline{\lambda}\gg 2a/\pi$ (maximum focusing). Then the ratio of the sin-functions in (2.10) possesses a very sharp maximum only for $\theta=\varphi=\pi/2$, given by (2.10a); this high intensity exists only within the very small angles (half-width angles)

$$\left.\begin{matrix}\Delta\theta\\ \Delta\varphi\end{matrix}\right\}=\pm 2\sqrt[4]{2}\,(\Omega a)^{-1/2}N^{-1/2} \qquad (2.12\text{a})$$

around $\theta=\varphi=\pi/2$ ("needle radiation"). Accordingly the integration of (2.10) can be restricted to this angular region. So we find for the total radiation power (see appendix):

$$L_{\mathrm{GW}}=2.8(MbA)^2\,\Omega^5 N/a$$
$$\triangleq 2.8\,G(\Omega^5/c^4)(MbA)^2 N/a\,. \qquad (2.12\text{b})$$

This result shows that because of the linear dependence of the right hand side of (2.12b) on N no superradiance for the total power seems to exist. On the other hand we have an astonishing dependence on frequency and light velocity of the form Ω^5/c^4, which is unusual for gravitation. This peculiar dependence is responsible for the fact that the radiation power (2.12b) is larger than the N-fold power of the single vibrator: Comparing (2.11b) and (2.12b) we obtain

$$\frac{L_{\mathrm{row}}}{N\,L_{\mathrm{vibrator}}}=2.8(15/16)\,\underline{\lambda}/a\gg 1\,. \qquad (2.13)$$

This means that a remnant of superradiance or a kind of stimulated emission is present.

3. Gravitational Radiation of the Diatomic Free Vibrating Finite Linear Chain

Since the calculation given above with the two-mass vibrators represents only a strong idealization of a laboratory experiment, the necessity exists to give a more realistic description and analysis for the single vibrator. In case of an experiment the vibrator would be realized practically by a thin piezoelectrical crystal; in this sense we consider in the following a diatomic linear chain as a useful model for such a solid and compare subsequently its radiation power with that of the vibrator. So the vibrator data can be fitted in such a way that the single vibrator represents appropriately a thin piezoelectrical crystal.

The total number of atoms of the chain may be $2N'$; this means we have N' atoms with mass M_1 and N' atoms with mass M_2, which lay alternately on a straight line (z-axis) separated by a distance a', so that the lattice parameter is $l=2a'$. Further we consider an harmonic interaction between next neighbours only described by the spring constant β. The Fig. 4 shows the chosen arrangement.

The equations of motion for the two sorts of masses are:

$$M_1\ddot{u}_1^s=-\beta(2u_1^s-u_2^s-u_2^{s+1}),$$
$$M_2\ddot{u}_2^s=-\beta(2u_2^s-u_1^s-u_1^{s-1}), \qquad (3.1)$$

where we have confined ourselves to the longitudinal vibrations $u_j{}^s$ in view of a reasonable comparison

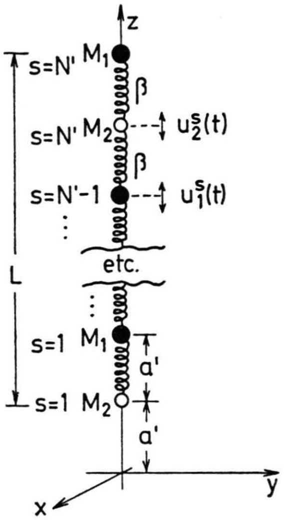

Fig. 4. The diatomic linear chain at the equilibrium position.

between chain and vibrator. The eigensolutions of (3.1) for the *free*-vibrating *finite* chain are treated in detail in [9]; they take the following normalized form:

$$u^s_{1m_i}(t) = B_{m_i} D_{m_i} A_1 \cos[2 s \varkappa_i a' - \Psi_1(m_i)] \cdot \cos[\Omega_{m_i} t + \gamma_{m_i}]$$

$$u^s_{2m_i}(t) = B_{m_i} D_{m_i} A_2 \cos[2 s \varkappa_i a' - \Psi_2(m_i)] \cdot \cos[\Omega_{m_i} t + \gamma_{m_i}] \tag{3.2}$$

with

$$D_{m_i} \equiv \sqrt{\frac{2(M_1 + M_2)}{(M_1 A_1^2 + M_2 A_2^2)}} \tag{3.2a}$$

and

$$\Psi_1(m_i) \equiv \varphi(m_i) = \operatorname{arc\,tg}\left[\frac{(A_1/A_2)_{m_i} - \cos\varkappa_i a'}{\sin\varkappa_i a'}\right];$$

$$\Psi_2(m_i) \equiv \varkappa_i a' + \varphi(m_i). \tag{3.2b}$$

The subindex m_i is the mode index ($m_i = 0, 1, 2, \ldots, (N'-1)$), wherein i stands for the acoustical branch ($i = 1$, $-$ sign) and the optical branch ($i = 2$, $+$ sign) of the well-known dispersion relation:

$$\Omega^2_{m_i} = \frac{\beta}{M_1 M_2}\{M_1 + M_2 \pm [M_1^2 + M_2^2 + 2 M_1 M_2 \cos(2\varkappa_i a')]^{1/2}\}. \tag{3.3}$$

For the ratio of the amplitudes A_1, A_2 in (3.2) one finds:

$$(A_1/A_2)_{m_i} = \frac{(2\beta/M_1)\cos(\varkappa_i a')}{(2\beta/M_1) - \Omega^2_{m_i}} = \frac{(2\beta/M_2) - \Omega^2_{m_i}}{(2\beta/M_2)\cos(\varkappa_i a')}, \tag{3.3a}$$

whereas the values of B_{m_i} and γ_{m_i} are determined by the initial conditions. From the free boundary condition, that means $u_1^0 = u_2^1$ and $u_1^{N'} = u_2^{N'+1}$ (no forces between the end atoms and external fictive atoms), it follows:

$$\varkappa_i = m_i \pi / 2 N' a',$$
$$m_i = 0, 1, 2, \ldots, (N'-1). \tag{3.4}$$

Next we calculate the gravitational radiation emission of the diatomic free-vibrating finite linear chain using (3.2) for the displacements out of the equilibrium position. For this we must use again the quadrupole approximation (1.4a) for the case $\lambda\!\!\!^{-} \gg 2N'a'$; otherwise the angular distribution of the radiation does not coincide with that of the vibrator. This means that only the low frequency modes of the chain, which also are piezoelectrically excited only, are usable for simulation of the vibrator behaviour*.

The mass distribution for the chain vibrating in the m_i-th mode is given by

$$\varrho(\boldsymbol{x}', t') = \sum_{s=1}^{N'}\{M_2\,\delta(\boldsymbol{x}' - [(2s-1)a' + u^s_{2m_i}(t')]\boldsymbol{e}_z) + M_1\,\delta(\boldsymbol{x}' - [2sa' + u^s_{1m_i}(t')]\boldsymbol{e}_z)\}. \tag{3.5}$$

Assuming that the displacements out of the equilibrium position are small, this means $|u^s_{2m_i}| \ll 2sa'$ and $|u^s_{1m_i}| \ll 2sa'$, we obtain for the radiation field (1.4a) after TT-projection according to (1.6)

$$h^{(m_i)\mathrm{TT}}_{jk} = \frac{(1 - n_z^2)}{r}\left\{2a' M_2 \sum_{s=1}^{N'}(2s-1)\,\ddot{u}^s_{2m_i} + 2a' M_1 \sum_{s=1}^{N'} 2s\,\ddot{u}^s_{1m_i}\right\} e_{jk}. \tag{3.6}$$

Inserting the time derivatives of (3.2) into (3.6) and carring out the sum we get finally for the radiation field

* The emission of gravitational radiation by the chain in the high frequency limit $\lambda\!\!\!^{-} \lesssim 2N'a'$ is treated in detail in [9].

$$h_{jk}^{(m_i)\mathrm{TT}}(\boldsymbol{x},t') = \begin{cases} 0 & \text{for } m_i \text{ even}, \\ \dfrac{B_{m_i} D_{m_i} a' \Omega_{m_i}^2}{r} (1 - n_z{}^2) \cos[\Omega_{m_i} t'] \dfrac{\cos[\varphi(m_i)]}{\sin^2[m_i \pi/4N']} \\ \quad \cdot \dfrac{\{M_2 A_2 \cos[m_i \pi/2N'] + M_1 A_1\}}{\{\cos[m_i \pi/2N'] + 1\}} e_{jk} & \text{for } m_i \text{ odd} \end{cases} \tag{3.7}$$

with the polarization tensor e_{jk} like in (2.9).

For the radial energy flux resulting from the m_i-th vibration mode we obtain with respect to (1.5):

$$T_{\mathrm{GW}}^{0r(m_i)} = \begin{cases} 0 & \text{for } m_i \text{ even}, \\ \dfrac{(B_{m_i} D_{m_i} a')^2}{32\pi r^2} \Omega_{m_i}^6 \dfrac{\cos^2[\varphi(m_i)]}{\sin^4[m_i \pi/4N']} \\ \quad \cdot \dfrac{\{M_2 A_2 \cos[m_i \pi/2N'] + M_1 A_1\}^2}{\{\cos[m_i \pi/2N'] + 1\}^2} (1 - n_z{}^2)^2 & \text{for } m_i \text{ odd}. \end{cases} \tag{3.8}$$

Evidently the angular distribution is identical with that of the vibrator (2.11a). For the Ω-dependence we have a Ω^6-law modified, however, by a form factor dependent on $m_i(\Omega)$. The integration over the total sphere gives the following expression for the energy loss due to gravitational waves:

$$L_{\mathrm{GW}}^{(m_i)} = \begin{cases} 0 & \text{for } m_i \text{ even}, \\ \dfrac{(B_{m_i} D_{m_i} a')^2}{15} \Omega_{m_i}^6 \dfrac{\cos^2[\varphi(m_i)]}{\sin^4[m_i \pi/4N']} \\ \quad \cdot \dfrac{\{M_2 A_2 \cos[m_i \pi/2N'] + M_1 A_1\}^2}{\{\cos[m_i \pi/2N'] + 1\}^2} & \text{for } m_i \text{ odd}. \end{cases} \tag{3.9}$$

Because only the low frequency modes are usable for simulation of the vibrator behaviour we restrict ourselves in the following to the acoustical modes with $m \ll N' (m = m_1)$. Then we obtain from (3.9) for the odd modes:

$$L_{\mathrm{GW}}^{(m)} = \tfrac{16}{15} (B_{m_i} D_{m_i} A_2)^2 \cdot [N'(M_1 + M_2)/2]^2 (N'a')^2 (2/m\pi)^4 \Omega_m^6. \tag{3.10}$$

The comparison with the vibrator formula should be performed in view of the experiment in such a way that the vibrator amplitude is identified with the amplitude between the ends of the chain. For this we find from (3.2) and (3.3a) in case of the acoustical branch with $m \ll N'$:

$$|C_m| \equiv \tfrac{1}{2} |(u_{1m}^{s=N'} - u_{2m}^{s=1})|_{t=0} = B_m D_m A_2. \tag{3.11}$$

Insertion into (3.10) yields:

$$L_{\mathrm{GW}}^{(m)} = \tfrac{16}{15} [N'(M_1 + M_2)/2]^2 \cdot (N'a')^2 (2/m\pi)^4 C_m^2 \Omega_m^6. \tag{3.12}$$

Now we compare the result (3.12) for the chain with that of the vibrator, formula (2.11b). We find that the vibrator simulates exactly a diatomic chain with respect to its gravitational radiation when the vibrator amplitude A is related with the amplitude C_m of the chain in the following way:

$$A = (2/m\pi)^2 C_m \tag{3.13}$$

for equal mass and length of chain and vibrator.

4. The Laboratory Experiment

Finally we turn our attenion to the question of proving the "needle radiation" obtained in Section 2. At first we substitute the vibrator data in the essential results of Sect. 2, formulae (2.10a) and (2.12b), by those of the diatomic chain representing a thin piezoelectrical crystal. With (3.13) and with the relations

$$M = M_c/2, \quad b = L_c/2 \tag{4.1}$$

(M_c, L_c mass and length of the chain) we find for the needle radiation and the total radiation power of a row of N piezoelectrical thin crystals (in CGS-units):

$$T_{\mathrm{GW}}^{0y} = \frac{G}{2\pi^5 c^5} \frac{M_c^2 L_c^2}{r^2} \frac{C_m^2}{m^4} \Omega_m^6 N^2 \tag{4.2}$$

and

$$L_{\mathrm{GW}} = \frac{2.8\,G}{\pi^4 c^4} M_c^2 L_c^2 \frac{C_m^2}{m^4} \Omega_m^5 N/a. \tag{4.3}$$

Because according to (3.3) $\Omega_m \sim m\sqrt{\beta}$ for the acoustical branch (with $m \ll N'$) one has to choose as high as possible acoustical vibration modes in the range $1 \ll m \ll N'$ for a promising experiment. For piezoelectrical excitations the corresponding maximum frequencies lay at $\Omega_m \simeq 10^9$ Hz $\triangleq \underline{\lambda}_m \simeq 30$ cm. According to realistic dispersion relations for quartz these frequencies are correlated with acoustical modes $\varkappa_m (= m\pi/2N'a') \simeq 10^4$ [11]. Choosing with respect to the magnitude of $\underline{\lambda}_m$ thin crystals of the length $L_c = 2N'a' = 10$ cm one finds for m the value $m \simeq 3 \times 10^4$.

Now we are able to estimate the magnitude of the radiation intensities (4.2) and (4.3) under best laboratory conditions. Taking $N = 10^4$ thin quartz crystals (length $L_c = 10$ cm, width 0.5 cm, mass $M_c = 5.5$ g) with a distance $a = 0.5$ cm, so that the crystals lay close together, we obtain a radiation power (its angular distribution is exactly that of Figure 3):

$$L_{GW} \simeq 2 \times 10^{-16} C_m^2 \text{ erg/sec}, \tag{4.4}$$

and at the end of the crystal row (length 50 m) an intensity:

$$T_{GW}^{0y} \simeq 7 \times 10^{-23} C_m^2 \text{ erg/sec cm}^2, \tag{4.5}$$

with C_m in cm. Under these conditions the half-width angles (2.12a) of the needle radiation amounts to $\pm 10^0$.

The maximum relative amplitudes for quartz lay in case of piezoelectric excitation approximately at 10^{-4} [12], so that the maximum values of C_m, attainable for the proposed experimental arrangement, are of the order of 10^{-3} cm. But even for amplitudes of this magnitude, the radiation power (4.4) and the radiation flux (4.5) may lay under the observational limit.

Appendix

We give here the determination of the half-width angles (2.12a) and the integration procedure for obtaining the total radiation power (2.12b) for the needle radiation of Section 2.

First we determine the half-width angle for the radiation flux of the vibrator row. Because formula (2.10) reaches noticeable values $\sim N^2$ only in a small neighbourhood of $\theta = \varphi = \pi/2$, when $\underline{\lambda} \gg 2a/\pi$, we set in (2.10) $\theta = \pi/2 + \delta$ and $\varphi = \pi/2 + \varepsilon$ with $\varepsilon \ll 1$, $\delta \ll 1$. Then we find from (2.10):

$$\frac{\sin^2\left[N\frac{\Omega a}{2}(\sin\theta\sin\varphi - 1)\right]}{\sin^2\left[\frac{\Omega a}{2}(\sin\theta\sin\varphi - 1)\right]}\sin^4\theta \cong \frac{\sin^2\left[N\frac{\Omega a}{4}(\varepsilon^2+\delta^2)\right]}{\sin^2\left[\frac{\Omega a}{4}(\varepsilon^2+\delta^2)\right]}. \tag{A.1}$$

With the auxiliary assumptions

$$\frac{\Omega a}{4}\varepsilon^2 \ll 1 \quad \text{and} \quad \frac{\Omega a}{4}\delta^2 \ll 1$$

it follows:

$$\frac{\sin^2\left[N\frac{\Omega a}{4}(\varepsilon^2+\delta^2)\right]}{\sin^2\left[\frac{\Omega a}{4}(\varepsilon^2+\delta^2)\right]} \cong \frac{\sin^2\left[N\frac{\Omega a}{4}(\varepsilon^2+\delta^2)\right]}{(\Omega a/4)^2(\varepsilon^2+\delta^2)^2}. \tag{A.2}$$

For the half-width angle $\varepsilon = \Delta\varphi$, $\delta = \Delta\theta$ the last expression must be equal to $\frac{1}{2}N^2$. So we find

$$\left.\begin{matrix}\Delta\varphi\\ \Delta\theta\end{matrix}\right\} = \frac{2\sqrt[4]{2}A^{1/2}}{(\Omega a)^{1/2}N^{1/2}}, \tag{A.3}$$

wherein A stands for a positive constant valued in the intervall $0 \leqq A \leqq 1$. Now we go back with (A.3) into (A.2) equated with $\frac{1}{2}N^2$ and get for A:

$$\sin\sqrt{2}A = A \Rightarrow A \simeq 1. \tag{A.4}$$

This leads to the result

$$\left.\begin{matrix}\Delta\varphi\\ \Delta\theta\end{matrix}\right\} = \frac{2\sqrt[4]{2}}{(\Omega a)^{1/2}N^{1/2}}. \tag{A.5}$$

Finally we prove the auxiliary conditions used above; with $\varepsilon, \delta = \Delta\varphi, \Delta\theta$ we find, using (A.5),

$$\left.\begin{matrix}\frac{\Omega a}{4}\varepsilon^2\\ \frac{\Omega a}{4}\delta^2\end{matrix}\right\} = \frac{\sqrt{2}}{N} \ll 1 \Leftrightarrow N \gg 1. \tag{A.6}$$

In case of a large number of vibrators in the row the auxiliary conditions are fulfilled automatically.

For the total radiation power we can write according to (2.10) under the conditions mentioned above, using (A.2),

$$L_{GW} = \int T^{0r}_{GW}\, r^2 \sin\theta \,\mathrm{d}\theta \,\mathrm{d}\varphi = \frac{1}{2\pi} (M b A)^2 \Omega^6 \cdot \int \frac{\sin^2\left[N \frac{\Omega a}{4} (\varepsilon^2 + \delta^2)\right]}{(\Omega a/4)^2 (\varepsilon^2 + \delta^2)^2} \,\mathrm{d}\varepsilon \,\mathrm{d}\delta . \tag{A.7}$$

With the substitution $\varepsilon = \hat{r} \cos\hat{\varphi}$, $\delta = \hat{r} \sin\hat{\varphi}$ we find

$$L_{GW} = \frac{1}{2\pi} (M b A)^2 \Omega^6 \int_0^{2\pi}\int_0^{\hat{r}_0} \frac{\sin^2\left[N \frac{\Omega a}{4} \hat{r}^2\right]}{(\Omega a/4)^2 \hat{r}^3} \,\mathrm{d}\hat{r}\,\mathrm{d}\hat{\varphi}$$

$$= (M b A)^2 \Omega^6 \int_0^{\hat{r}_0} \frac{\sin^2\left[N \frac{\Omega a}{4} \hat{r}^2\right]}{(\Omega a/4)^2 \hat{r}^3} \,\mathrm{d}\hat{r} , \tag{A.8}$$

wherein $\hat{r}_0$ corresponds to the half-width angles (A.5) and may be determined for simplicity in such a way that the integrand of (A.8) vanishes; thus it follows

$$\hat{r}_0 = \frac{2\sqrt{\pi}}{\sqrt{\Omega a}} N^{-1/2} . \tag{A.9}$$

Then by partial integration and the substitution $x = N(\Omega a/2)\hat{r}^2$ $(0 \leqq x \leqq 2\pi)$ we get

$$L_{GW} = \frac{2}{\Omega a} (M b A)^2 \Omega^6 N \int_0^{2\pi} \frac{\sin x}{x} \,\mathrm{d}x . \tag{A.10}$$

Because the value of the sin-integral is 1.42 [13], we obtain finally

$$L_{GW} = 2.8 (M b A)^2 \Omega^5 N/a . \tag{A.11}$$

[1] For a review of Weber's experiments, see J. Weber in B. Bertotti, ed., "Proceedings of Course 56 of the International School of Physics "Enrico Fermi" " (Academic, New York, 1973); see also V. Trimble and J. Weber, Ann. N. Y. Acad. Sci. **224**, 93 (1973).
[2] U. Kh. Kopvillem and V. R. Nagibarov, JETP Lett. **2**, No. 12, 329 (1965).
[3] L. E. Halpern and R. Desbrandes, Ann. Inst. H. Poincaré **XI** No. 3, 309 (1969).
[4] F. Sacchetti and D. Trevese, Nuov. Cim. **35B**, N. 1, 61 (1976).
[5] J. Weber and G. Hinds, Dept. of Phys. and Astron. Univ. of Maryland Tech. Report No. 634.
[6] G. Schäfer and H. Dehnen, Phys. Rev. D **23**, 2129 (1981).
[7] L. D. Landau and E. M. Lifshitz, The Classical Theory of Fields, 4th Ed., Pergamon Press, Oxford 1975, §110.
[8] C. W. Misner, K. S. Thorne, and J. A. Wheeler, Gravitation, Freeman, San Francisco 1973, p. 992.
[9] F. Romero B., Lineare Gitterstrukturen als Emitter und als Absorber von Gravitationsstrahlung, Doctoral Dissertation Univ. Konstanz 1981.
[10] H. C. Ohanian, Gravitation and Spacetime, W. W. Norton, New York 1976.
[11] H. Bilz and W. Kress, Phonon Dispersion Relations in Insulators, Springer-Verlag, Berlin 1979.
[12] Verbal information by H. Bömmel. See also J. Van Randeraat and R. E. Setterington, (Eds.) Piezoelectric Ceramics, 2nd. Ed., Mullard Ltd., London 1974, Chapter 2.
[13] See e. g. Jahnke-Emde-Lösch, Tafeln höherer Funktionen, 7th Ed., Teubner, Stuttgart 1966, p. 24.